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Twice Scattered Particles in a Plane Are Uniformly Distributed

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Abstract It was recently shown (Physica A 216:299–315, 1995) that in two dimensions the sum of three vectors each of whose lengths is exponentially distributed, whose direction is uniformly distributed and such that the sum of their lengths is l, is uniformly distributed on a disk of radius l. We state here this random walk result in terms of scattering of particles as follows: in two dimensions twice isotropically scattered particles by random (i.e., Poisson distributed) scatterers are uniformly distributed. We show that there is no other dimension d and no other number of scatterings s for which the corresponding result (i.e., uniform distribution on a d-dimensional sphere after s scatterings) holds.

Keywords Continuous time random walk · Multiple scattering

1 Outlook

Let there be a particle which in an infinitesimal interval of time dt may undergo isotropic scattering with probability λdt. Suppose, further, that the scatterers are randomly (i.e., Poisson) distributed and that their masses are much larger than the mass of the particle, whose speed, therefore, does not change. In the limit in which the interaction with the scatterer is infinitely short ranged the trajectory of the particle is a broken line for which the length x of its segments is distributed with a pdf $\lambda e^{-\lambda x}$ and whose directions are uniformly distributed [1, 2]. Recently M. Franceschetti [2] has shown that in two dimensions the sum of three vectors whose length is exponentially distributed, whose direction is uniformly distributed and such that the sum of their lengths is l, is uniformly distributed on a disk of radius l. In view of the third sentence of this paragraph this is equivalent to the following statement: let there be in two dimensions an isotropic front $\frac{\delta(r-vt)}{2\pi vt}$ of such particles which leaves a point with speed v (that is, the particles would all remain uniformly distributed on a circumference

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of radius vt if they didn't scatter). Then those particles which have been scattered exactly twice are uniformly distributed inside a disk whose radius is vt.

This result is unexpected. It is almost intuitive, however (a "word derivation" is given in [2]), that in 1 dimension the once scattered particles are uniformly distributed *inside* a segment of length $2vt$, while on the end points of this segment there are Dirac deltas which correspond to the particles that have kept their direction after the scattering. M. Franceschetti therefore examined the natural conjecture that d times scattered particles in d dimensions are uniformly distributed inside a d-dimensional sphere and showed by means of a calculation involving the second moment that this was not true. But the question of whether there is uniform distribution inside a d-dimensional sphere of s times scattered particles for some other (s,d) pairs remained unanswered. We show here that this happens only in the above mentioned two cases, that is (1,1) and (2,2).

2 The Proof

In 1 dimension this sort of problems can be tackled with the telegraph equation [3], a differential equation. In two or more dimensions one has to solve an integral equation [4, 5] or do space-time convolution of the spherically expanding front of particles [1, 2]. Neither of these two calculations is easy. However, the even radial moments of the distribution of s times scattered particles are given in [1], and this will be enough to prove the negative result claimed at the end of the preceding section.

The expected value of r^m, where r is the radial coordinate of a homogeneous d-dimensional sphere of radius R, is (see, e.g., Appendices A and C of reference [1]):

$$\langle r^m \rangle_{d,sphere} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{R^{d+m}}{d+m}. \tag{1}$$

To obtain the corresponding quantity for an expanding spherical shell of particles which have scattered s times in d dimensions we adapt formulae (24) and (25) of reference [1]:

$$\langle r^{2m} \rangle_{d,s\,scatterings} = (2m)!\ R^{2m} \left[\frac{\Gamma(d/2)}{\Gamma(1/2)}\right]^s \frac{s!}{(2m+s)!}\, \sigma(m,d,s) \tag{2}$$

and

$$\sigma(m,d,s) = \sum_{i_1+\cdots+i_{s+1}=m} \frac{\Gamma(m+d/2)}{\Gamma(i_1+d/2)\cdots\Gamma(i_{s+1}+d/2)} \frac{\Gamma(i_1+1/2)\cdots\Gamma(i_{s+1}+1/2)}{\Gamma(m+1/2)}. \tag{3}$$

The combinatorial function σ is in general quite lengthy to compute, because the restriction under the summation is obviously related to the number of partitions of an integer n, which grows as $\frac{1}{4\sqrt{3}\,n} e^{2\pi\sqrt{\frac{n}{6}}}$ ([6], p. 56). But for small values of m it is still tractable. $\sigma(1,d,s)$ has $s+1$ terms, all of them corresponding to the different realizations of the decomposition $1 = 1+0+\cdots+0$. $\sigma(2,d,s)$ has $s+1$ terms which correspond to the different realizations of the decomposition $2 = 2+0+\cdots+0$ and $\binom{s+1}{2}$ terms which correspond to the different realizations of the decomposition $2 = 1+1+0+\cdots+0$.

In reference [2] the following necessary condition for the distribution to be uniform is explored:

$$\langle r^2 \rangle_{d,sphere} = \langle r^2 \rangle_{d,s\,scatterings}, \tag{4}$$

which yields:

$$s = \frac{2(d+2)}{d} - 2. \tag{5}$$

Besides the two already known cases this condition is also satisfied by $(s, d) = (1, 4)$.

To settle the question we consider the ratios:

$$\frac{\langle r^4 \rangle_{d,sphere}}{R^2\ \langle r^2 \rangle_{d,sphere}} = \frac{d+2}{d+4} \tag{6}$$

and:

$$\frac{\langle r^4 \rangle_{d,scatterings}}{R^2\ \langle r^2 \rangle_{d,scatterings}} = \frac{4!\,\sigma(2,d,s)\ 4!}{6!\,\sigma(1,d,s)\ 2!} = \frac{2}{5}\ \frac{\sigma(2,d,s)}{\sigma(1,d,s)} = \frac{2}{5}\left(1 + \frac{(2+d)\ s}{6\ d}\right). \tag{7}$$

We equate them and obtain:

$$\frac{d+2}{d+4} = \frac{2}{5}\left(1 + \frac{(2+d)\ s}{6\ d}\right), \tag{8}$$

from which we can solve for d:

$$d = \frac{3(s-1) \pm \sqrt{9(s-1)^2 + 8s(9-s)}}{9-s}. \tag{9}$$

When $s > 9$, $\sqrt{9(s-1)^2 + 8s(9-s)} < 3(s-1)$ and the denominator is positive while the denominator is negative. Therefore $d < 0$. When $s = 9$, d is not defined. Then we need only to check $s \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. The positive (s, d) solutions are $\{(1, 1), (2, 2), (3, 3.236), (4, 4.9), (5, 7.36), (6, 11.4), (7, 19.44), (8, 43.47)\}$. We see then, as advertised, that this necessary condition for the conjecture to hold is satisfied only by the two already known cases.

The proof that the already known cases are the only ones for which the conjecture holds is now complete, but we are still going to address two concerns that the reader might have. One is that the lhs of (8) was obtained (formula (6)) considering a distribution which is uniform inside a sphere, zero outside the sphere and with no measure-theoretic singularities (i.e., Dirac deltas) on the boundary. Equation (8), however, is also satisfied by the pair $(s, d) = (1, 1)$, for which the distribution is $1/2$ in the open segment (−R, R) plus $\frac{\delta(r-R)+\delta(r-R)}{4}$ [2]. It just turns out that for this distribution the ratio (6) holds. Remember that the logic of the proof is that a *necessary* condition is not satisfied by any other (s, d) pair. The other concern is that there might be cases other than $(s, d) = (1, 1)$ for which the distribution after s scatterings is uniform *inside* the sphere and singular on its boundary and for which (8) does not hold (as it does for $(s, d) = (1, 1)$). But a little reflection shows that there are no other cases for which the measure supported by the shell of the sphere is not zero. For a particle to reach the boundary of the sphere it must keep its course after each scattering. The probability of this happening is zero. This argument only breaks down in the already studied 1-dimensional case, in which the number of available directions is not infinite but 2.

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